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# A polyhedral study of the diameter constrained minimum spanning tree problem

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## ABSTRACT

This paper provides a first polyhedral study of the diameter constrained minimum spanning tree problem (DMSTP). We introduce a new set of inequalities, the circular-jump inequalities which strengthen the well-known jump inequalities. These inequalities are further generalized in two ways: either by increasing the number of partitions defining a jump, or by combining jumps with cutsets. Most of the proposed new inequalities are shown to define facets of the DMSTP polytope under very mild conditions. Currently best known lower bounds for the DMSTP are obtained from an extended formulation on a layered graph using the concept of central nodes/edges. A subset of the new families of inequalities is shown to be not implied by this layered graph formulation.

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## 1. Introduction

Given a graph  $G = (V, E)$ , with  $n = |V|$  nodes and  $m = |E|$  edges, a spanning tree of  $G$ , denoted by  $T = (V, E_T)$  is a connected subgraph of  $G$  without cycles. The diameter of the graph  $G$  is the length of the longest shortest path between any two nodes in  $G$  (in this paper we consider the length of a path to be its number of edges). The diameter constrained minimum spanning tree problem (DMSTP) is defined as follows: Given a graph  $G$  with edge costs  $c_e \geq 0$ , for all  $e \in E$ , and a diameter limit  $D$ , the goal is to identify a minimum cost spanning tree of  $G$  whose diameter does not exceed  $D$ . The DMSTP is typically used to model decentralized network design applications in which all nodes need to communicate with each other at minimum cost, while making sure that certain quality of service constraints are met. Thereby, the DMSTP is especially appropriate for networks in which node processing times dominate over the latencies on edges, so that having fewer intermediary nodes along a communication path implies a lower delay, in general. Hence, quality of service can be measured through the number of nodes traversed along each communication path and diameter constraints can be used to guarantee a certain network performance with respect to availability or reliability (see, e.g. [1,19]).

The most successful method to solve DMSTP is based on graph concepts related with centers in trees and sophisticated extended formulations based on layered graphs that are tailored for network design problems with length constraints (see [12]). For some time it was not known how to describe inequalities that guarantee the length of the paths using only edge variables. The work by Dahl [3] has made a significant contribution in this area by proposing the so-called jump constraints to model length-constrained paths. More recently, a complete characterization of jump inequalities for the hop-constrained shortest path problem has been provided by Riedl [18]. Jump constraints have been also considered in the

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context of the hop constrained trees (Dahl et al. [4,6]) and more general network design problems with hop constraints and survivability requirements (see, e.g., Bendali et al. [2], Dahl et al. [7] and Huygens et al. [13,14]). The latter works involve deep polyhedral studies where, among other results, these inequalities (or small adaptations obtained by changing the right-hand side due to the survivability requirements) have been shown to be facet defining under some mild conditions (see, e.g., Bendali et al. [2], Diarrassouba et al. [9] and Huygens et al. [14]).

*A generic edge-based ILP formulation.* In order to contextualize our study we start by introducing a generic formulation (1)–(5) for the problem, considering undirected edge design variables  $x_e \in \{0, 1\}$ , for all  $e \in E$ , which indicate whether edge  $e$  is used in the solution.

$$\min \sum_{e \in E} c_e x_e \quad (1)$$

$$x(E(S)) \leq |S| - 1 \quad \forall S \subset V, |S| \geq 3 \quad (2)$$

$$x(E) = |V| - 1 \quad (3)$$

$$\mathbf{x} \in \mathcal{F}_D \quad (4)$$

$$\mathbf{x} \in \{0, 1\}^{|E|} \quad (5)$$

In this formulation  $x(M)$ ,  $M \subseteq E$ , stands for  $\sum_{e \in M} x_e$ , and  $E(S)$  represents the edges with both endpoints in  $S \subset V$ . Constraints (2) are the subtour elimination constraints, that together with Eq. (3) guarantee that the obtained solution is a spanning tree. Let  $\mathcal{G}(\mathbf{x})$  denote the subgraph of  $G$  induced by the edges  $e \in E$ , such that  $x_e = 1$ , and let  $P_{\mathcal{G}(\mathbf{x})}(u, v)$  denote a shortest path in this subgraph between  $u \in V$  and  $v \in V$ . The set  $\mathcal{F}_D$  is defined as  $\mathcal{F}_D = \{\mathbf{x} \in \{0, 1\}^{|E|} \mid \text{for each } u, v \in V, u \neq v, \exists P_{\mathcal{G}(\mathbf{x})}(u, v) : |P_{\mathcal{G}(\mathbf{x})}(u, v)| \leq D\}$ . Hence,  $\mathcal{F}_D$  is the set of incidence vectors such that the induced subgraph contains a feasible path between any two nodes, i.e., a path of length at most  $D$ .

*Scientific contribution and outline.* In Section 2, we first recall jump inequalities and show that they do not define facets of the DMSTP polytope which is in contrast to many other hop-constrained network design problems. Hence, the main question addressed by the present article is how to derive an ILP formulation in the natural space of edge-variables that relies on strong and (ideally) facet defining inequalities. We show that this can be done by considering a new family of facet defining *circular-jump inequalities* which are specific for the DMSTP. They imply the jump inequalities, and define facets of the underlying polytope under mild conditions. So-called *generalized-circular-jump inequalities* are introduced in Section 3 for which we also give conditions under which they are facet defining. In Section 4, we then show how the previously introduced inequalities can be combined with a cutset inequality, resulting in a new valid and (in most cases) facet defining inequality. In Section 5 we study packing-type inequalities that provide an alternative way to define a natural space formulation for the DMSTP. Section 6 contains a theoretical comparison of the circular-jump inequalities to the strongest formulation known from the literature. We show that they are not implied by this layered graph formulation. Finally, conclusions are drawn in Section 7 where we also discuss open questions that can be addressed in future research.

## 2. Circular-jump inequalities

In this section we first establish the dimension of the DMSTP polytope, and summarize necessary notation. We then recall the definition of jump inequalities [3], and show that, in contrast to many other hop-constrained network design problems for which they are facet defining, this does not hold for the DMSTP. In order to overcome this drawback, we introduce a new family of stronger *circular-jump inequalities* that ensure a valid ILP formulation in the space of edge-variables. We also provide conditions under which these inequalities are facet defining.

*Dimension of the polytope and notation.* Let  $\mathcal{P}$  be the convex hull of all DMSTP feasible solutions, i.e.,

$$\mathcal{P} = \text{conv}\{\mathbf{x} \in \{0, 1\}^{|E|} \mid \mathbf{x} \text{ satisfies (2)–(4)}\}.$$

We first analyze the dimension of the DMSTP polytope,  $\dim(\mathcal{P})$ , given by the following theorem whose proof is given in Appendix A. Throughout this paper we assume that  $G$  is a complete graph.

**Theorem 1.** *The following results hold:*

1. For  $D = 2$ ,  $\dim(\mathcal{P}) = n - 1$ .
2. For  $D \geq 3$ ,  $\dim(\mathcal{P}) = m - 1$ .

In the remainder of this article, we will consider different non-trivial partitions  $(J_0, J_1, \dots, J_{D+l})$  of the node set  $V$  into  $D+l+1$ ,  $l \geq 1$ , disjoint, nonempty subsets  $J_i$ ,  $0 \leq i \leq D+l$ , i.e.,  $V = \bigcup_{i=0}^{D+l} J_i$ ,  $J_i \cap J_j = \emptyset$ ,  $0 \leq i < j \leq D+l$ . For brevity we will also use notation  $[P, S]$  to refer to the set of edges between node sets  $P, S \subset V$ , i.e.,  $[P, S] = \{[u, v] \in E \mid u \in P, v \in S\}$ .

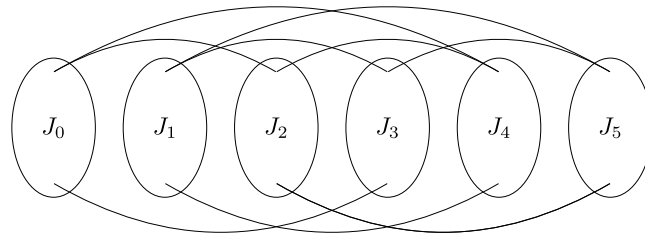


Fig. 1. Circular jump  $C_J$  for a given partition  $(J_0, J_1, \dots, J_5)$ .  $D = 4$ .

## 2.1. Jump inequalities

Jump inequalities were originally proposed by Dahl [3] in the context of the undirected hop constrained shortest path problem. The so-called  $(s, t, D)$ -jump inequalities, used to model the constrained paths are defined as follows: Let  $(J_0, J_1, \dots, J_{D+1})$ , be a non-trivial partition of the node set  $V$  into  $D + 2$  subsets such that  $s \in J_0$  and  $t \in J_{D+1}$ . The set of edges  $J = \bigcup_{0 \leq i < j-1 \leq D} [J_i, J_j] \subset E$  is a jump. The associated jump inequality is  $x(J) \geq 1$ . The explanation why these inequalities prevent paths with more than  $D$  edges, is as follows (see, e.g., Dahl [3]): Suppose that there exists a solution such that it contains no edge of the given jump  $J$ . Since the graph induced by this solution must be connected, there must exist a path starting at a node in subset  $J_0$ , passing through nodes from all subsets from  $J_1$  to  $J_D$  and ending at a node in subset  $J_{D+1}$ . This path has length at least  $D + 1$  and thus the solution cannot be feasible. The jump inequalities can be adapted in a straightforward way to model constrained shortest paths in the context of the DMSTP. Let  $\mathcal{U}$  be the set of all possible jumps on  $G$  (regarding all possible partitions into  $D + 2$  nonempty subsets), then the jump constraints for the DMSTP are defined as follows:

$$x(J) \geq 1, \quad J \in \mathcal{U}. \quad (J)$$

We obtain a valid formulation for the DMSTP by using these inequalities in place of  $\mathcal{F}_D$  in the generic description (1)–(5) given above. Note that it is sufficient to consider only those jumps where  $|J_0| = |J_{D+1}| = 1$  and that those jumps dominate the remaining ones. For the undirected shortest path problem with at most  $D$  edges, jump inequalities can be facet defining (see, Dahl [3]). As we shall show in the next section, this is not the case for the DMSTP.

## 2.2. Circular-jump inequalities

Let  $(J_0, J_1, \dots, J_{D+1})$  be a non-trivial partition of  $V$  as described above. Then,  $C_J = \bigcup_{0 \leq i < j-1 \leq D} [J_i, J_j] \setminus [J_0, J_{D+1}]$  is a circular jump. Note that the difference between a circular jump  $C_J$  and a jump  $J$  is that the former one does not contain the edges connecting the first set with the last one. Observe that, e.g., the circular jumps defined by the partitions  $(J_0, J_1, \dots, J_{D+1})$ ,  $(J_1, J_2, \dots, J_{D+1}, J_0)$ ,  $\dots$ ,  $(J_D, J_{D+1}, J_0, \dots, J_{D-1})$ , as well as  $(J_0, J_{D+1}, J_D, \dots, J_1)$  are identical. Thus, while in a jump we have ordered subsets with a given first subset and a given last subset, in a circular jump there is no first set neither last set and thus, the designation. Fig. 1 illustrates the set of edges of a circular jump for a graph with  $n \geq 6$  and  $D = 4$ . Let  $\mathcal{J}$  be the set of all circular jumps, then the corresponding circular-jump constraints are defined as follows:

$$x(C_J) \geq 1, \quad C_J \in \mathcal{J}. \quad (cJ)$$

Note, that based on the previous observation when comparing a jump inequality with a circular-jump inequality, it is clear that each jump inequality (J) is dominated by a circular-jump inequality (cJ) obtained from the same partition. Note also that a circular-jump inequality, dominates several jump inequalities (by choosing any subset of the circular jump to be the first set of the jump inequality and keeping the same order for the remaining subsets).

**Proposition 1.** Circular-jump inequalities (cJ) are valid for the DMSTP polytope  $\mathcal{P}$ .

**Proof.** Circular-jump inequalities obviously forbid paths with more than  $D$  edges. To show that these inequalities do not cut off feasible solutions, let us assume the opposite, i.e., there exists a feasible solution  $T$  that violates one (cJ) inequality which is specified by a partition  $(\tilde{J}_0, \tilde{J}_1, \dots, \tilde{J}_{D+1})$ . Since we have seen that jump inequalities are valid, this (cJ) inequality is violated because an edge  $\{i_0, i_{D+1}\}$  with  $i_0 \in \tilde{J}_0$  and  $i_{D+1} \in \tilde{J}_{D+1}$  belongs to the solution. However, since the graph induced by the solution must be connected, there exists  $k$ ,  $0 < k < D + 1$  such that there is a path from node  $i_0$  to a node  $i_k$ , passing through all subsets  $\tilde{J}_0$  to  $\tilde{J}_k$  and a path from node  $i_{D+1}$  to a node  $i_{k+1}$ , passing through subsets  $\tilde{J}_{D+1}, \tilde{J}_D, \tilde{J}_{D-1}, \dots, \tilde{J}_{k+1}$ . Since no jump edges belong to this solution, the length of this path is  $\geq D + 1$  and hence the solution  $T$  is infeasible, which is a contradiction.  $\square$

Hence, we obtain another valid formulation for the DMSTP by using the inequalities (cj) in place of  $\mathcal{F}_D$  in the generic description (1)–(5) given above. The validity of the circular-jump inequalities for the DMSTP depends on the fact that the underlying solution is a spanning tree. Contrary to the jump inequalities (J), they are not valid for other related problems with hop constraints such as, e.g., the hop constrained shortest path problem. Also in contrast to the jump inequalities (J) the circular-jump inequalities (cj) define facets of  $\mathcal{P}$  for  $D \geq 4$ . Before proving this result, we observe that the DMSTP can be solved in polynomial time for  $D = 2, 3$ . In case of  $D = 2$ , all feasible solutions are stars, and the following result characterizes the circular-jump inequalities that are facet defining.

**Theorem 2.** For  $D = 2$ , circular-jump inequalities define facets of  $\mathcal{P}$  if and only if at most one set of the partition contains more than one node.

**Proof.** See Appendix B.  $\square$

In the rest of the paper we will concentrate on the more general case, when  $D \geq 3$ . The proofs of the following two results are given in Appendix B.

**Theorem 3.** For  $D = 3$ , circular-jump inequalities (cj) define facets of  $\mathcal{P}$  if two consecutive sets of the partition contain exactly one node.

**Theorem 4.** For  $D \geq 4$ , circular-jump inequalities (cj) define facets of  $\mathcal{P}$ .

Before showing how to generalize these inequalities (see next sections), we observe that from a given infeasible solution we may derive several violated (cj) inequalities. Consider a spanning tree solution with at least one path being too long and for the moment let us assume that the length of this path is  $D + 1$ . The  $D + 2$  nodes of this path determine the “seeds” of the  $D + 2$  subsets in the partition defining a circular jump. There are two intuitive strategies for assigning the remaining nodes to the subsets of the partition that have been called path and layered approach in the context of classical jump constraints in the literature [4]. One strategy is to assign all nodes of a subtree rooted at a seed node to the subset seeded by that node. The other is to pick one extreme node of the path as a root of a tree that is directed away from that node. Then, the distance of each node to the root will define its partition subset. Typically, the longest path of an infeasible solution will be much longer than  $D + 1$ . In Section 3 we propose a more general class of circular-jump inequalities that considers more than  $D + 1$  subsets (and thus allow to assign nodes of infeasible paths with length strictly greater than  $D + 1$  to different subsets). This generalization is based on ideas proposed by Dahl and Gouveia [5] for generalizing the jump constraints for the hop-constrained shortest path problem. In Section 4 we introduce further generalizations that can be viewed as combining a cutset inequality with (generalized) circular-jump inequalities defined on a subset of nodes.

### 3. Generalized-circular-jump inequalities

In this section we introduce the family of *generalized circular jumps* that can be seen as a generalization of circular jumps, for the case that the number of sets in a partition is larger than  $D + 2$ . Each generalized-circular-jump inequality may have edge coefficients greater than one and inequalities defined by  $k + 1$  sets can be obtained through Chvátal–Gomory rounding from inequalities defined by  $k$  sets (and in particular, inequalities with  $D + 3$  sets can be obtained from (cj)). We show that generalized-circular-jump inequalities are also facets of the DMSTP polytope. For  $k \in \mathbb{N}$ , let  $P = (J_0, J_1, \dots, J_{D+k})$  be a non-trivial partition of the node set  $V$ . Let, furthermore

$$C_\ell(P) = \bigcup_{i=0}^{D+k} [J_i, J_{(i+\ell+1) \bmod (D+k+1)}],$$

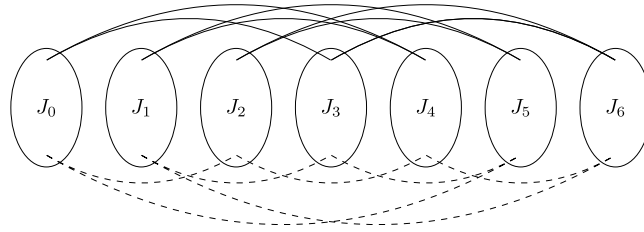
for each  $\ell \in \{1, \dots, k-1\}$  denote the set of edges that jump over *exactly*  $\ell$  consecutive sets of the partition and

$$C_k(P) = \left( \bigcup_{0 \leq i < j-1 \leq D+k-1} [J_i, J_j] \right) \setminus \bigcup_{\ell'=1}^{k-1} C_{\ell'}(P)$$

be the set of edges that jump over *at least*  $k$  sets. Denoting by  $\mathcal{J}^k$  the set of non-trivial partitions of  $V$  into  $D + k + 1$  sets, the family of generalized-circular-jump inequalities (gcj) is given as

$$\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) \geq k, \quad P \in \mathcal{J}^k, k \in \{1, 2, \dots, D-1\}. \quad (\text{gcj})$$

We will use notation  $k$ -(gcj) to refer to the subset of all (gcj) for a particular  $k \in \{1, 2, \dots, D-1\}$ . Clearly, for  $k = 1$ , we obtain the standard circular-jump inequality (cj). To give some intuition on these inequalities, consider such an inequality with  $D + 3$  subsets (i.e., with  $k = 2$ ). The inequality states that any feasible solution needs to use at least one edge that jumps over two subsets or at least two edges that jump over exactly one subset. In the general case (with right-hand



**Fig. 2.** Illustration of a generalized-circular-jump inequality for a given partition  $(J_0, J_1, \dots, J_6)$ ,  $k = 2$ , and  $D = 4$ . Solid edges above the partitions jump over at least two subsets (and thus the corresponding variables have coefficients equal to two) while dashed edges below the partitions skip a single subset (and the corresponding variables have coefficients of one).

side equal to any  $k$  such that  $1 \leq k \leq D - 1$ ) any feasible solution has to use at least  $k$  edges that jump over exactly one subset (i.e.,  $k$  edges from  $C_1(P)$ ), or at least one edge that jumps over at least  $k$  subsets (i.e., an edge from  $C_k(P)$ ), or a combination of edges from  $\bigcup_{\ell=1}^{k-1} C_\ell(P)$  so that in total at least  $k$  subsets are jumped over.

Fig. 2 illustrates a generalized-circular-jump inequality for  $D = 4$  and  $k = 2$  which is formally given as:

$$\begin{aligned}
 & x(J_0, J_2) + 2x(J_0, J_3) & + 2x(J_0, J_4) + x(J_0, J_5) & & & & \\
 & \quad + x(J_1, J_3) & + 2x(J_1, J_4) + 2x(J_1, J_5) & & + x(J_1, J_6) & & \\
 & & + x(J_2, J_4) + 2x(J_2, J_5) & & + 2x(J_2, J_6) & & \\
 & & \quad + x(J_3, J_5) & & + 2x(J_3, J_6) & & \\
 & & & & + x(J_4, J_6) \geq 2. & & 
 \end{aligned}$$

As shown in the proof of the following result, the validity of generalized-circular-jump inequalities follows from a Chvátal–Gomory rounding argument:

**Proposition 2.** Generalized-circular-jump inequalities (gcj) are valid for the DMSTP polytope  $\mathcal{P}$ .

**Proof.** We will prove the theorem by induction using the fact that inequalities (gcj) are the standard circular-jump inequalities when  $k = 1$  as the induction starting step. Assuming that the inequalities are valid for  $k - 1$ , we will use a Chvátal–Gomory rounding argument to prove their validity for  $k$ . Let  $P = (J_0, J_1, \dots, J_{D+k})$  be the partition associated to a  $k$ -(gcj). To derive the corresponding inequality, we consider  $D + k + 1$  generalized-circular-jump inequalities with right-hand side equal to  $k - 1$  associated with the following partitions:

$$\begin{aligned}
 P_1 &= \{J_0 \cup J_1, J_2, \dots, J_{D+k}\} \\
 P_2 &= \{J_0, J_1 \cup J_2, J_3, \dots, J_{D+k}\} \\
 &\dots \\
 P_{D+k} &= \{J_0, \dots, J_{D+k-2}, J_{D+k-1} \cup J_{D+k}\} \\
 P_{D+k+1} &= \{J_1, \dots, J_{D+k-1}, J_{D+k} \cup J_0\}
 \end{aligned}$$

Consider an edge  $e \in C_\ell(P)$ . As will be shown in the following, the coefficient obtained from summing over all  $D + k + 1$  inequalities corresponding to partitions  $P_1, \dots, P_{D+k+1}$  is given by

$$\xi_\ell = \begin{cases} \ell(D + k) - 1 & \text{if } \ell < k, \\ (k - 1)(D + k + 1) & \text{if } \ell \geq k. \end{cases}$$

In the following, without loss of generality (due to circularity) we assume that  $e = \{u, v\}$  with  $u \in J_0$  and  $v \in J_{\ell+1}$ .

For  $\ell < k$  we observe that the coefficient associated to edge  $e$  is equal to  $\ell - 1$  for the inequalities corresponding to the  $\ell + 1$  partitions  $P_1, \dots, P_{\ell+1}$  and equal to  $\ell$  for the inequalities associated to the remaining  $D + k - \ell$  partitions  $P_{\ell+2}, \dots, P_{D+k+1}$ . Overall, we obtain  $(\ell - 1)(\ell + 1) + \ell \cdot (D + k - \ell) = \ell(D + k) - 1$ . For  $\ell \geq k$ , the coefficient associated to edge  $e$  is equal to  $k - 1$  for the inequalities corresponding to all  $D + k + 1$  partition, also giving the claimed overall value of  $(k - 1)(D + k + 1)$ .

Since the smallest coefficient is  $\xi_1 = D + k - 1$ , we obtain

$$\left\lceil \frac{\xi_\ell}{\xi_1 + 1} \right\rceil = \begin{cases} \ell & \text{if } \ell < k, \\ k & \text{otherwise.} \end{cases}$$

Finally, since the obtained right-hand side is  $(D + k + 1)(k - 1)$  we obtain the  $k$ -(gcj) by dividing through  $\xi_1 + 1$  and rounding up all coefficients on the left- and right-hand side, respectively.  $\square$

Not only are the generalized-circular-jump inequalities valid for the DMSTP, they also define facets of the underlying polytope. The proof of the following result can be found in Appendix C.

**Theorem 5.** For  $D \geq 4$  and  $1 \leq k \leq D - 2$ , generalized-circular-jump inequalities (gcj) define facets of  $\mathcal{P}$ .

#### 4. Cut-jump inequalities

In this section, we derive new families of inequalities by associating nodes of an infeasible path of length strictly greater than  $D$  to a (generalized) circular jump on a subset of nodes (in contrast to the complete set of nodes), and combining them with the remaining nodes using a cutset. In these constraints, notation  $x(R, S)$ ,  $R, S \subseteq V$ , stands for  $\sum_{e \in [R, S]} x_e$ .

**Proposition 3.** For  $1 \leq k \leq D - 1$  let  $(R, S)$  be a non-trivial partition of the set of nodes  $V$  such that  $|R| \geq D + k + 1$ . Let, furthermore  $P = (J_0, J_1, \dots, J_{D+k})$  be a non-trivial partition of  $R$  into  $D + k + 1$  sets,  $C_\ell(P)$ ,  $1 \leq \ell < k$ , be the edge set jumping over exactly  $\ell$  sets from  $P$ ,  $C_k(P)$  be the edge set jumping over at least  $k$  sets from  $P$ , and  $\mathcal{J}^k(R)$  be the set of all such partitions of  $R$ . Then, the following cut generalized-circular-jump inequalities

$$\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) + kx(R, S) + kx(E(S)) \geq k \cdot (|S| + 1),$$

$$R \subset V, |R| \geq D + k + 1, S = V \setminus R, P \in \mathcal{J}^k(R) \quad (\text{cgcj})$$

are valid for the DMSTP polytope  $\mathcal{P}$ .

**Proof.** To see the validity of these constraints assume that  $x(E(S)) = q$  (observe that  $q \leq |S| - 1$ ). Hence, the subgraph induced by  $S$  is composed of  $|S| - q$  (connected) components. Each of these components needs to be connected to  $R$ , that is  $x(R, S) \geq |S| - q$ . If  $x(R, S) > |S| - q$  then the inequality is obviously valid. If  $x(R, S) = |S| - q$ , then  $\sum_{\ell=1}^k \ell x(C_\ell(P)) \geq k$  must be satisfied (otherwise we will have an infeasible path in the partition defined by  $R$ ).  $\square$

**Theorem 6.** For  $4 \leq D \leq n - k - 2$  and  $1 \leq k \leq D - 3$ , cut generalized-circular-jump inequalities (cgcj) define facets of  $\mathcal{P}$ .

**Proof.** See Appendix D.  $\square$

#### 5. Rounded circuit packing inequalities

For many network design problems there are two equivalent ways of expressing natural space formulations. Either by using so-called cut inequalities that in some sense guarantee connectivity of the solutions or by using so-called packing inequalities stating that feasible solutions cannot have more than a certain number of edges in given subsets. One example of this is given by formulations for the spanning tree problem (see, e.g., [15]) and the Steiner tree problem (see, e.g., [10]). Several routing problems can also be modeled in these two ways such as, e.g., the well-known traveling salesman problem [8]. Usually, the two approaches are shown to be equivalent (or more, precisely, to lead to formulations with equivalent LP relaxations) since one model can be transformed into the other by using equalities that are included in the models. Up to now, the inequalities that have been discussed for modeling “path length constraints” in the DMSTP are cut-type inequalities. In this section, we briefly discuss a set of packing-type inequalities. These *circuit packing* inequalities are given as

$$x(C) \leq |C| - \left\lceil \frac{|C|}{D} \right\rceil, \quad \text{for all cycles } C \subset E, D + 2 \leq |C| \leq |V|. \quad (6)$$

We will show that using them in place of  $\mathcal{F}_D$  yields a valid DMSTP formulation and that they are related, via equality (3), to some of the inequalities that have been discussed before.

Circuit packing constraints (6) state that for any cycle  $C \subset E$  with at least  $D + 2$  edges, the number of edges in any feasible solution is bounded from above by  $|C| - \lceil |C|/D \rceil$ . A similar set of inequalities is presented in Gouveia [11] for the directed hop-constrained minimum spanning tree problem (see also [17]). Proposition 4 shows that circuit packing inequalities associated to cycles of length  $D + 2$  are equivalent to some of the inequalities studied in the previous section. This result will be used to show the validity of circuit packing constraints in the general case, cf. Proposition 5. Notice, that the first part of the proof of Proposition 5 also implies that (for complete input graphs) it suffices to consider the circuit packing constraints associated to cycles of length  $D + 2$  in place of  $\mathcal{F}_D$  in order to obtain a valid formulation for the DMSTP.

**Proposition 4.** Let  $C$  be a cycle in  $G$  of length  $D + 2$  such that  $C = \{i, i + 1 \} \bmod (D + 2), 0 \leq i \leq D + 1\}$ . Then, a rounded circuit packing inequality induced by  $C$  corresponds to:

1. a circular-jump constraint (cj), if  $|V| = D + 2$ .
2. a cut generalized-circular-jump constraint (cgcj) with  $k = 1$ , if  $|V| > D + 2$ .



**Proof.**

1. Let  $V = \{0, \dots, D+1\}$  and let  $CJ$  be the circular-jump corresponding to the partition  $(\{0\}, \{1\}, \dots, \{D+1\})$  in which each set is a single node from  $V$  following the ordering of nodes from  $C$ . By subtracting the rounded circuit packing inequality implied by  $C$  from  $x(E) = |V| - 1$ , we obtain a circular-jump constraint  $x(E) - x(C) = x(CJ) \geq 1$ .
2. For  $k = 1$ , let  $P = (\{0\}, \{1\}, \dots, \{D+1\})$  be the partition of node set  $R = \{0, 1, \dots, D+1\}$  and  $S = V \setminus R$ . We observe that using the same transformation as above we obtain a cut generalized-circular-jump constraint  $x(E) - x(C) = x(C_1(P)) + x(E(S)) + x(R, S) \geq 1 + |S|$ .  $\square$

**Proposition 5.** Replacing  $x \in \mathcal{F}_D$  in the generic formulation (2)–(5) by (6) gives a valid model for the DMSTP.

**Proof.** We first show, that any infeasible solution is cut off by the given inequalities. Assume that there exists an integer solution feasible to the generic model with constraints (6) instead of (4), such that, without loss of generality, it contains a path of length  $D+1$ . Let  $V_C$  be the set of nodes of this path. Obviously  $|V_C| = D+2$ . By joining the first and the last node of the path with an edge, we obtain a cycle  $C$  (this is possible, since  $G$  is complete), and for this cycle, we notice that (6) is violated. The value of the left-hand side equals  $D+1$ , while the value of the right-hand side is strictly less than  $D+1$ , which is a contradiction.

It remains to show that inequalities (6) are valid for the DMSTP polytope  $\mathcal{P}$ . This result will be shown by induction over  $|C|$  using the result from Proposition 4 as start. Thus, assume that circuit-packing inequalities are valid for all cycles of length at most  $|C|$ . Further assume, that there exists a spanning tree  $T = (V, E_T)$  of diameter at most  $D$  that violates a circuit packing inequality for a cycle  $C' \subset E$  of length  $|C'| = |C| + 1$ . Thus,  $x(C') \geq 1 + |C'| - \lceil |C'|/D \rceil$ .

First assume that there exists some  $l \in \mathbb{N}$  such that  $|C'| = lD + 1$  and notice that  $l \geq 2$  since  $|C'| \geq D+2$  holds by definition of (6). Then, the right-hand sides of the circuit packing constraints (6) for  $C'$  and for a cycle  $C$  of length  $|C| = |C'| - 1$  are identical. Thus,  $x(C') \geq 1 + |C| - \lceil |C|/D \rceil = 1 + l(D-1)$ . Therefore,  $|C'| - x(C') \leq lD + 1 - (lD + 1 - l) = l$ , i.e., at most  $l$  edges of  $C'$  at not in  $T$ . If no path of  $T$  that lies on  $C'$  has length greater than  $D$ , this is only possible if  $C' \cap E_T$  contains a path of length  $D$  and at least one path of length at least  $D-1$ . Any spanning tree containing two such paths would, however, violate the diameter limit as connecting two such paths would always induce a path of length at least  $D+1$ .

Hence,  $l \in \mathbb{N}$  such that  $|C'| = lD + 1$  cannot exist which implies that the right hand sides of the circuit packing constraints (6) associated to  $C'$  and a cycle containing one edge less differ by one. Let  $C' = \{u_0, u_1\}, \{u_1, u_2\}, \dots, \{u_k, u_0\}$  and for each  $u_i$ ,  $0 \leq i \leq k$ ,  $\tilde{C}[u_i]$  be the cycle of length  $|C'| - 1$  obtained from  $C'$  after removing the two edges adjacent to  $u_i$  and adding the edge  $\{u_{i-1}, u_{i+1}\}$  (with appropriate modulo calculations for indexes if  $i = 0$  or  $i = k$ ). Thus, we have  $x(C') \geq 1 + |C'| - \lceil |C'|/D \rceil = 2 + |\tilde{C}[u_i]| - \lceil |\tilde{C}[u_i]|/D \rceil$  and  $x(\tilde{C}[u_i] \setminus \{u_{i-1}, u_{i+1}\}) \leq x(\tilde{C}[u_i]) \leq |\tilde{C}[u_i]| - \lceil |\tilde{C}[u_i]|/D \rceil$  since inequalities (6) hold by assumption for all cycles shorter than  $C'$ . These two inequalities imply  $x_{u_{i-1}, u_i} + x_{u_i, u_{i+1}} \geq 2$ , i.e., both edges of  $C'$  adjacent to  $u_i$  must be included in  $T$ . Repeating this calculation for each node  $u_i$ ,  $0 \leq i \leq k$  on  $C'$  implies that all edges of  $C'$  must be included in  $T$  which contradicts the fact that  $T$  is a spanning tree of  $G$ .  $\square$

## 6. Theoretical comparison to the layered graph approach

As noted before, the most efficient method for solving the DMSTP is still the one given in Gouveia et al. [12]. The underlying formulation models hop-constraints with layered graphs which we briefly recall next. For simplicity, we discuss only the case for  $D$  even. Essentially Gouveia et al. [12] used the fact that the DMSTP can be modeled as a special hop-constrained minimum spanning tree problem, where the root of the tree is appropriately selected and showed that the DMSTP can be reformulated as a Steiner arborescence problem with few additional constraints on an appropriately defined layered graph which implicitly ensures that the diameter of a solution is at most  $D$ . For the case of  $D$  even, this graph  $G_L = (V_L, A_L)$  is defined as  $V_L = \{r\} \cup \{i_h \mid i \in V, 0 \leq h \leq D/2\}$  and  $A_L = \{(i_h, j_{h+1}) \mid \{i, j\} \in E, 0 \leq h \leq D/2 - 1\} \cup \{(r, i_0) \mid i \in V\}$ . Using arc decision variables  $a_{ij} \in \{0, 1\}$ , for all  $(i, j) \in A = \{(i, j) \mid \{i, j\} \in E\}$ , and layered arc design variables  $X_{ij}^h \in \{0, 1\}$ , for all  $(i_{h-1}, j_h) \in A_L$ , the layered-graph formulation for the DMSTP is given by (7)–(12) where  $X[M]$ ,  $M \subseteq A_L$ , stands for  $\sum_{(i_{h-1}, j_h) \in M} X_{ij}^h$ .

$$\min \sum_{(i,j) \in A} c_{ij} a_{ij} \quad (7)$$

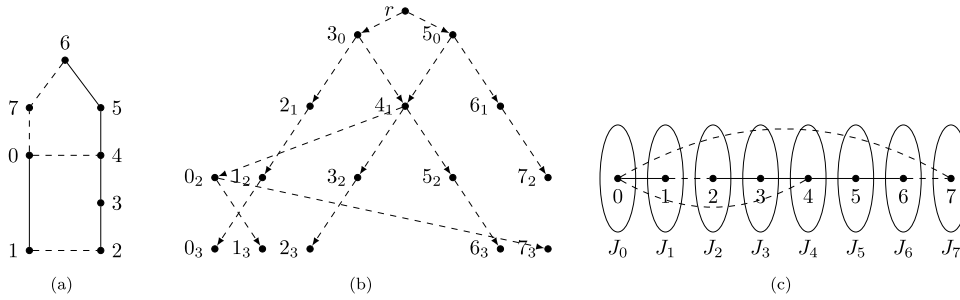
$$\text{s.t. } X[\delta^-(W)] \geq 1 \quad \forall W \subset V_L \setminus \{r\}, \exists i \in V : \{i_h \mid 0 \leq h \leq D/2\} \subseteq W \quad (8)$$

$$X[\delta^+(r)] = 1 \quad (9)$$

$$\sum_{h=0}^{D/2} X[\delta^-(i_h)] = 1 \quad \forall i \in V \quad (10)$$

$$a_{ij} = \sum_{h=1}^{D/2-1} X_{ij}^h \quad \forall (i, j) \in A \quad (11)$$





**Fig. 3.** (a) Optimal LP-solution of the layered graph model for  $D = 6$  (projected back into the space of  $\mathbf{x}$  variables). (b) Optimal LP-solution of the layered graph model for this instance. (c) Partition showing that this solution violates the circular jump constraints. Recall, that the edge  $\{0, 7\}$  between the first and the last partition  $J_0$  and  $J_7$ , respectively, is not contained in the circular jump. Dashed edges (arcs) indicate LP values of 0.5, solid edges (arcs) of 1.

$$\mathbf{X} \geq \mathbf{0}, \mathbf{a} \in \{0, 1\}^{|\mathbf{A}|} \quad (12)$$

This model has been shown to theoretically dominate all previously proposed ones. A branch-and-cut algorithm developed from this model is the current state-of-the-art for solving DMSTP instances to proven optimality. Although the linear programming relaxation of (7)–(12) is integral for most of the instances considered in [12], we show in the following that it does not imply most of the inequalities introduced in this article.

Let  $v_{LP}(LG)$  denote the value of the LP-relaxation of the layered graph model, and let  $v_{LP}(CJ)$  be the value of the LP-relaxation of the model defined by the tree constraints (2) and (3) and the circular-jump inequalities (cJ). The following result shows that the lower bound of the layered graph model can be as worse as  $1/2$  of the bound obtained by our model involving only circular-jump inequalities.

**Proposition 6.** *There exist DMSTP instances such that*

$$\frac{v_{LP}(LG)}{v_{LP}(CJ)} \leq \frac{1}{2}.$$

**Proof.** Consider a complete graph with node set  $V = \{0, 1, \dots, 7\}$  with edge costs  $c_{ij} = 0$  for all  $0 \leq i \leq 7$  and  $j = (i + 1) \bmod 7$ ,  $c_{04} = M$  and  $c_e = \infty$  for all the remaining edges. For  $D = 6$  and a sufficiently large  $M \in \mathbb{N}$ , the optimal LP-solution of the layered graph model is given in Fig. 3(b) and its value is  $v_{LP}(LG) = 6.5 \cdot 0 + M/2 = M/2$ . Projected back in the space of  $\mathbf{x}$  variables, this fractional solution is shown in Fig. 3(a). Let  $CJ$  be the circular jump obtained from the partition  $\{\{0\}, \{1\}, \dots, \{7\}\}$ . Then this solution violates the corresponding circular-jump inequality since  $x(CJ) = 0.5 < 1$ . The optimal LP-solution of the ILP model derived from the circular-jump inequalities is  $v_{LP}(CJ) = 6 \cdot 0 + M = M$ , so we obtain  $\frac{v_{LP}(LG)}{v_{LP}(CJ)} = \frac{1}{2}$ .  $\square$

Let  $\text{proj}_x(\mathcal{P}(LG))$  be the polytope of the LP-relaxation of the layered graph formulation projected to the natural space of undirected edge design variables. We conclude that  $\text{proj}_x(\mathcal{P}(LG))$  may contain solutions that violate some of the inequalities introduced in this article. On the other hand, computational experiments showed that it is not too difficult to find exemplary solutions that satisfy all natural space inequalities proposed in this article, but are not contained in  $\text{proj}_x(\mathcal{P}(LG))$ . Thus, neither of the associated polytopes is contained in the other.

## 7. Conclusions

In this article, we performed the first study of natural space formulations for the diameter constrained minimum spanning tree problem (DMSTP) which use variables associated to undirected edges only. We proposed different classes of inequalities generalizing the concept of jump inequalities [3], which have been used in the literature for other problems with hop constraints. For most of the new inequalities, we showed that they define facets of the DMSTP polytope under very mild conditions. Validity of (generalized) circular-jump inequalities relies on the fact that the underlying problem is a spanning tree. In contrast, the idea that led to cut-jump inequalities (i.e., jumps on subsets of nodes) does not rely on this fact. Thus, it can possibly be used to define generalizations of the jump inequalities (instead of (generalized-) circular-jump inequalities) for other problems as well. We believe, that the resulting “cut-jump” inequalities are worth to be analyzed for other network design problems with length constraints. We also show that circular-jump inequalities are not implied by current state-of-the-art ILP model based on a layered graph reformulation [12].

This work lays down foundations for some future investigations including: (i) The identification of additional valid inequalities in the natural space and their relevance for obtaining tight LP relaxation bounds; (ii) The development of efficient (heuristic) separation routines for the proposed inequalities in order to make a corresponding branch-and-cut

approach applicable to very large scale instances which cannot be treated by the layered graph approach; (iii) Derivation of a compact (layered graph?) reformulation that implies the inequalities studied in this work. We are particularly interested in finding a compact model whose linear programming relaxation implies the circular-jump constraints. Recall that a compact model that contains standard s-t-jump inequalities is known from the literature [6,12]. An appropriate mixed integer linear programming formulation which contains jump-constraints for the DMSTP is obtained by the intersection of hop constrained spanning trees, each with a root on a different node and such that distance from any node to the root node is at most  $D$  (we skip this part here but refer the reader to Dahl et al. [6] and Gouveia et al. [12] for further details). While we can show that the strongest such model does not imply the circular-jump constraints, our attempts to provide a “similar” model implying the circular-jump constraints have failed. (iv) A better understanding of the projection of layered graph models to the natural space of design variables (which applies to any problem that is modeled in this way, not only the DMSTP).

### CRedit authorship contribution statement

**Luis Gouveia:** Conceptualization, Methodology, Validation, Writing - original draft, Writing - review & editing. **Markus Leitner:** Conceptualization, Methodology, Validation, Investigation, Writing - original draft, Writing - review & editing. **Ivana Ljubić:** Conceptualization, Methodology, Validation, Writing - original draft, Writing - review & editing.

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### Appendix A. Dimension proofs

**Proof of Theorem 1.** We first show that  $\dim(\mathcal{P}) = n - 1$  if  $D = 2$ . Recall that the set of feasible solutions to the DMSTP with  $D = 2$  is given by the  $n$  “stars”  $S^i = (V, E_i)$  with  $E_i = \{\{i, j\} \mid \forall j \in V \setminus \{i\}\}$ , for all  $i \in V$ . By looking at the matrix in which each of the incidence vectors represents its row, we easily observe that this matrix has a full rank, which is  $n$ .

To show that  $\dim(\mathcal{P}) = m - 1$  holds for  $D \geq 3$ , we first note that the dimension of the DMSTP polytope cannot be larger than  $m - 1$  since all feasible solutions satisfy equation (3). To show that the dimension is at least  $m - 1$  we construct a set  $\Psi$  of  $m = \binom{n}{2}$  solutions whose edge-incidence vectors are affinely independent. For  $i, j \in V$ ,  $i < j$ , let  $S_{i,j}^k = (V, E_k \setminus \{k, i\} \cup \{i, j\})$ ,  $E_k = \{\{k, i\} \mid \forall i \in V \setminus \{k\}\}$  be the graph obtained from the star  $S^k = (V, E_k)$  after attaching node  $i$  to node  $j$  instead of node  $k$ . Note that each graph  $S_{i,j}^k$  is a feasible solution since its diameter is precisely three. To simplify notation, we will also use  $S^i$ ,  $i = 1, 2, \dots, n - 1$ , and  $S_{k+1,j}^k$ ,  $k = 1, \dots, n - 2$ , to refer to the incidence vectors of the corresponding solutions. In the following, we will also use notation  $(S)_{ij}$  to refer to the entry of a vector  $S$  associated to edge  $\{i, j\} \in E$ .

For a complete graph with  $n$  nodes, we consider set  $\Psi$  consisting of stars  $S^i$ ,  $i = 1, 2, \dots, n - 1$ , and solutions  $S_{k+1,j}^k$ ,  $k = 1, \dots, n - 2$ ,  $j = k + 2, \dots, n$ , see Table 1 for an illustration. Notice that  $|\Psi| = (n - 1) + \sum_{i=2}^{n-1} (n - i) = \sum_{i=1}^{n-1} i = \binom{n}{2} = m$ . We will prove that the  $m - 1$  vectors in  $\mathcal{T} = \{S - S^1 \mid S \in \Psi \setminus \{S^1\}\}$  illustrated in Table 2 are linearly independent. The latter result implies that the  $m$  incidence vectors in the set  $\Psi$  are affinely independent and thus that  $\dim(\mathcal{P}) = m - 1$  if  $D \geq 3$ .

For  $k \in \{1, 2, \dots, n - 2\}$ , define  $\mathcal{T}_k \subseteq \mathcal{T}$  as  $\mathcal{T}_k = \{(S_{i+1,j}^i - S^1) \mid i = 1, \dots, k, j = i + 2, \dots, n\} \cup \{(S^i - S^1) \mid i = 2, \dots, k + 1\}$ . From Table 2 it is easy to observe that the set of vectors in  $\mathcal{T}^1 = \{S_{2,i}^1 - S^1 \mid i = 3, \dots, n\} \cup \{S^2 - S^1\}$  is linearly independent. We will show that linear independence of the vectors in  $\mathcal{T}_{k-1}$ ,  $k \in \{2, \dots, n - 2\}$  implies linear independence of the vectors in  $\mathcal{T}_k$ . Since  $\mathcal{T}_{n-2} = \mathcal{T}$ , linear independence of all vectors in  $\mathcal{T}$  follows.

Assume that the vectors in  $\mathcal{T}_{k-1}$ ,  $k \in \{2, \dots, n - 2\}$  are linearly independent and consider the set of vectors  $\mathcal{T}_k = \mathcal{T}_{k-1} \cup \{S_{k+1,j}^k - S^1 \mid j = k + 1, \dots, n\} \cup \{S^{k+1} - S^1\}$ . We observe that  $(S_{k+1,j}^k - S^1)_{k+1,j} = 1$  for each  $j \in \{k + 2, \dots, n\}$  while  $(S)_{k+1,j} = 0$  holds for all  $S \in \mathcal{T}_k \setminus \{(S_{k+1,j}^k - S^1, S^{k+1} - S^1)\}$ . Thus linear independence of  $\mathcal{T}_{k-1}$  implies linear independence of  $\mathcal{T}_k \setminus \{S^{k+1} - S^1\}$ .

It remains to show that  $(S^{k+1} - S^1)$  cannot be expressed as a linear combination of the remaining vectors in  $\mathcal{T}^k$ . To see that this is impossible, we focus on the entries of all vectors in  $\mathcal{T}^k$  associated to edges  $\{1, k + 1\}$  and  $\{1, n\}$  and observe that  $(S)_{1,k+1} = (S)_{1,n}$  for all  $S \in \mathcal{T}^k \setminus \{(S^{k+1} - S^1)\}$ . Thus, the entries associated to these two edges in any linear combination of vectors from  $\mathcal{T}^k \setminus \{(S^{k+1} - S^1)\}$  must be identical. Since  $(S^{k+1} - S^1)_{1,k+1} = 0$  and  $(S^{k+1} - S^1)_{1,n} = -1$  we conclude that it  $(S^{k+1} - S^1)$  cannot be expressed as linear combination of the remaining vectors from  $\mathcal{T}^k$  and, thus, that  $\mathcal{T}^k$  is linearly independent.

**Table 1**  
Illustration of set  $\Psi$  containing  $m$  affinely independent vectors.

	$x_{12}$	$x_{13}$	$x_{14}$	...	$x_{1,n-2}$	$x_{1,n-1}$	$x_{1n}$	$x_{23}$	$x_{24}$	$x_{25}$	...	$x_{2,n-1}$	$x_{2n}$	$x_{34}$	$x_{35}$	...	$x_{3,n-1}$	$x_{3n}$	$x_{45}$	$x_{46}$	...	$x_{4,n-1}$	$x_{4n}$	...	$x_{n-2,n-1}$	$x_{n-2,n}$	$x_{n-1,n}$
$S^1$	1	1	1	...	1	1	1	0	0	0	...	0	0	0	0	...	0	0	0	0	...	0	0	...	0	0	0
$S^2$	1	0	0	...	0	0	0	1	1	1	...	1	1	0	0	...	0	0	0	0	...	0	0	...	0	0	0
$S^3$	0	1	0	...	0	0	0	1	0	0	...	0	0	1	1	...	1	1	0	0	...	0	0	...	0	0	0
$S^4$	0	0	1	...	0	0	0	0	1	0	...	0	0	1	0	...	0	0	1	1	...	1	1	...	0	0	0
...				...							...					...					...			...			
$S^{n-1}$	0	0	0	...	0	1	0	0	0	0	...	1	0	0	0	...	1	0	0	0	...	1	0	...	1	0	1
$S^1_{23}$	0	1	1	...	1	1	1	0	0	...	0	0	0	0	...	0	0	0	0	...	0	0	...	0	...	0	0
$S^1_{24}$	0	1	1	...	1	1	0	1	0	...	0	0	0	0	...	0	0	0	0	...	0	0	...	0	...	0	0
...				...						...					...					...			...				
$S^1_{2n}$	0	1	1	...	1	1	0	0	0	...	0	1	0	0	...	0	0	0	0	...	0	0	...	0	...	0	0
$S^2_{34}$	1	0	0	...	0	0	0	1	1	...	1	1	1	0	...	0	0	0	0	...	0	0	...	0	...	0	0
$S^2_{35}$	1	0	0	...	0	0	0	1	1	...	1	1	0	1	...	0	0	0	0	...	0	0	...	0	...	0	0
...				...						...					...					...			...				
$S^2_{3n}$	1	0	0	...	0	0	0	1	1	...	1	1	0	0	...	0	1	0	0	...	0	0	...	0	...	0	0
$S^3_{45}$	0	1	0	...	0	0	0	1	0	0	...	0	0	1	...	1	1	1	0	...	0	0	...	0	...	0	0
$S^3_{46}$	0	1	0	...	0	0	0	1	0	0	...	0	0	1	...	1	1	0	1	...	0	0	...	0	...	0	0
...				...						...					...					...			...				
$S^3_{4n}$	0	1	0	...	0	0	0	1	0	0	...	0	0	1	...	1	1	0	0	...	0	1	...	0	...	0	0
...				...						...					...					...			...				
$S^{n-2}_{n-1,n}$	0	0	0	...	1	0	0	0	0	0	...	0	0	0	...	0	0	0	0	...	0	0	...	0	...	0	1

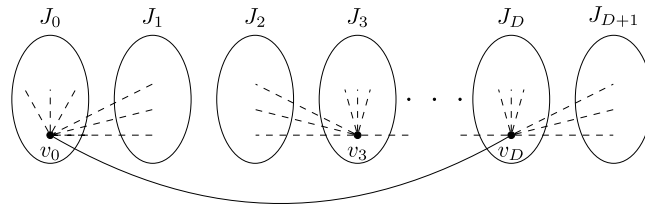
**Table 2**  
Illustration of set  $\mathcal{T}$  containing  $m-1$  linearly independent vectors.

	$x_{12}$	$x_{13}$	$x_{14}$	$\dots$	$x_{1,n-2}$	$x_{1,n-1}$	$x_{1n}$	$x_{23}$	$x_{24}$	$x_{25}$	$\dots$	$x_{2,n-1}$	$x_{2n}$	$x_{34}$	$x_{35}$	$\dots$	$x_{3,n-1}$	$x_{3n}$	$x_{45}$	$x_{46}$	$\dots$	$x_{4,n-1}$	$x_{4n}$	$\dots$	$x_{n-2,n-1}$	$x_{n-2,n}$	$x_{n-1,n}$
$S_{23}^1 - S^1$	-1	0	0	$\dots$	0	0	0	1	0	0	$\dots$	0	0	0	0	$\dots$	0	0	0	0	$\dots$	0	0	$\dots$	0	0	0
$S_{24}^1 - S^1$	-1	0	0	$\dots$	0	0	0	0	1	0	$\dots$	0	0	0	0	$\dots$	0	0	0	0	$\dots$	0	0	$\dots$	0	0	0
$\dots$				$\dots$							$\dots$					$\dots$					$\dots$			$\dots$			$\dots$
$S_{2n}^1 - S^1$	-1	0	0	$\dots$	0	0	0	0	0	0	$\dots$	0	1	0	0	$\dots$	0	0	0	0	$\dots$	0	0	$\dots$	0	0	0
$S^2 - S^1$	0	-1	-1	$\dots$	-1	-1	-1	1	1	1	$\dots$	1	1	0	0	$\dots$	0	0	0	0	$\dots$	0	0	$\dots$	0	0	0
$S_{34}^2 - S^1$	0	-1	-1	$\dots$	-1	-1	-1	0	1	1	$\dots$	1	1	1	0	$\dots$	0	0	0	0	$\dots$	0	0	$\dots$	0	0	0
$S_{35}^2 - S^1$	0	-1	-1	$\dots$	-1	-1	-1	0	1	1	$\dots$	1	0	1	1	$\dots$	0	0	0	0	$\dots$	0	0	$\dots$	0	0	0
$\dots$				$\dots$							$\dots$					$\dots$					$\dots$			$\dots$			$\dots$
$S_{3n}^2 - S^1$	0	-1	-1	$\dots$	-1	-1	-1	0	1	1	$\dots$	1	0	0	0	$\dots$	0	1	0	0	$\dots$	0	0	$\dots$	0	0	0
$S^3 - S^1$	-1	0	-1	$\dots$	-1	-1	-1	1	0	0	$\dots$	0	0	1	1	$\dots$	1	1	0	0	$\dots$	0	0	$\dots$	0	0	0
$S_{45}^3 - S^1$	-1	0	-1	$\dots$	-1	-1	-1	1	0	0	$\dots$	0	0	1	$\dots$	1	1	1	0	$\dots$	0	0	$\dots$	0	0	0	
$S_{46}^3 - S^1$	-1	0	-1	$\dots$	-1	-1	-1	1	0	0	$\dots$	0	0	1	$\dots$	1	1	0	1	$\dots$	0	0	$\dots$	0	0	0	
$\dots$				$\dots$							$\dots$					$\dots$					$\dots$			$\dots$			$\dots$
$S_{4n}^3 - S^1$	-1	0	-1	$\dots$	-1	-1	-1	1	0	0	$\dots$	0	0	1	$\dots$	1	1	0	0	$\dots$	0	1	$\dots$	0	0	0	
$S^4 - S^1$	-1	-1	0	$\dots$	-1	-1	-1	0	1	0	$\dots$	0	0	1	0	$\dots$	0	0	1	1	$\dots$	1	1	$\dots$	0	0	0
$\dots$				$\dots$							$\dots$					$\dots$					$\dots$			$\dots$			$\dots$
$S_{n-1,n}^{n-2} - S^1$	-1	-1	-1	$\dots$	0	-1	-1	0	0	0	$\dots$	0	0	0	0	$\dots$	0	0	0	0	$\dots$	0	0	$\dots$	0	1	1
$S^{n-1} - S^1$	-1	-1	-1	$\dots$	-1	0	-1	0	0	0	$\dots$	1	0	0	0	$\dots$	1	0	0	0	$\dots$	1	0	$\dots$	1	0	1

## Appendix B. Facet proofs for circular-jump inequalities

Given a circular jump determined by a partition  $(J_0, \dots, J_{D+1})$ , to simplify the notation and avoid case distinction, we refer to subsets  $J_i$  of a given jump for any  $i \in \mathbb{Z}$  implicitly assuming an appropriate modulo calculation  $i \bmod (D+2)$  whenever  $i < 0$  or  $i > D+1$ . In general, for a partition  $(J_0, \dots, J_p)$ , when we state  $J_i$ , we refer to  $J_{i \bmod (p+1)}$ . Furthermore, for  $v_i \in J_i$ ,  $0 \leq i \leq D+1$ , we introduce the following notation:

- $I(v_i) = \{\{v_i, j\} \mid j \in J_i, j \neq v_i\}$ ,  $v_i \in J_i$ , i.e.,  $I(v_i)$  is the edge set of a star containing all nodes of  $J_i$  with center  $v_i$  ("in-edges")
- $B(v_i) = \{\{j, v_i\} \mid j \in J_{i-1}\}$ ,  $v_i \in J_i$ , i.e.,  $B(v_i)$  is a set of edges connecting all nodes from  $J_{i-1}$  to one particular node  $v_i$  from  $J_i$ . ("back-edges")
- $F(v_i) = \{\{v_i, j\} \mid j \in J_{i+1}\}$ ,  $v_i \in J_i$ , i.e.,  $F(v_i)$  is a set of edges connecting all nodes from  $J_{i+1}$  to one particular node  $v_i$  from  $J_i$  ("forward-edges")

Fig. 4. Solution  $T^{0,D}$ .

Similarly, let  $I(J_i) = \bigcup_{v_i \in J_i} I(v_i)$  be all inner edges of  $J_i$ , for  $0 \leq i \leq D+1$ , and notice that edges between two consecutive subsets  $J_i$  and  $J_{i+1}$  can be represented as  $[J_i, J_{i+1}] = B(J_{i+1}) = F(J_i)$  where  $B(J_i) = \bigcup_{v_i \in J_i} B(v_i)$  and  $F(J_i) = \bigcup_{v_i \in J_i} F(v_i)$ .

Let  $v_i \in J_i$ ,  $0 \leq i \leq D+1$ , be an arbitrarily chosen node, and  $CJ$  be a given circular jump determined by a partition  $(J_0, \dots, J_{D+1})$ . Let  $T^{p,q}$  denote a feasible solution using exactly one jump edge  $\{v_p, v_q\} \in CJ$  where  $q - p \geq 2$ ,  $p, q \in \{0, \dots, D+1\}$ . For example, Fig. 4 illustrates a solution  $T^{0,D}$  that contains edges from  $I(v_0) \cup F(v_0) \cup B(v_3) \cup F(v_D)$ , a path  $\{v_3, v_4\}, \dots, \{v_{D-1}, v_D\}$  and sets  $I(v_i)$ , i.e., stars centered at each  $v_i$ ,  $i = 3, \dots, D$ .

**Proof of Theorem 2.** First note that since for  $D = 2$  we have  $\dim(\mathcal{P}) = n - 1$ , and each feasible solution is a star, the face  $F$  corresponding to a circular-jump inequality must contain the incidence vectors of all but one feasible solutions to be a facet. We now show that each inequality such that all but one sets in the partition are singletons is facet defining. Without loss of generality assume that  $(J_0, J_1, J_2, J_3) = (\{1\}, \{2\}, \{3\}, \{4, \dots, n\})$ . Then, the corresponding circular-jump inequality is given by

$$x(CJ) = x_{13} + \sum_{i=4}^n x_{2i} \geq 1 \quad (13)$$

We observe that the support of (13) contains  $n - 3$  edges incident to node 2 and one edge incident to any other node  $i \in V \setminus \{2\}$ . Hence, each star centered at a node  $i \neq 2$  is a feasible DMSTP solution that satisfies (13) with equality. Thus, we have  $\dim(\mathcal{P}) = n - 1$  affinely independent points.

To show the converse, assume that at least two sets of partition  $(J_0, J_1, J_2, J_3)$  contain two or more nodes. Without loss of generality (due to circularity), we assume that  $J_3$  is one of them and  $J_i$  (for some  $i \in \{0, 1, 2\}$ ) is the second one. Then, the number of edges incident to a node  $j \in J_1$  that are in the support of the corresponding circular-jump inequality is at least two and thus all points corresponding to graphs with center node  $j \in J_1$  cannot be contained in  $F$ . Thus, the inequality can only define a facet if  $|J_1| = 1$  holds. Thus,  $J_0$  or  $J_2$  contains at least two nodes and there must exist a node  $v$  with two or more incident edges that are in the support of the corresponding circular-jump inequality. Thus, all points corresponding to such graphs cannot lie on  $F$ . Using a similar line of argument, we obtain that  $J_0$  and  $J_1$  must be singletons, which contradicts our initial assumption.

**Proof of Theorem 3.** We now show that for  $D = 3$ , circular-jump inequalities define facets of  $\mathcal{P}$  if two consecutive sets of the partition contain exactly one node. Without loss of generality, we assume that  $J_0 = \{v_0\}$  and  $J_4 = \{v_4\}$  are two consecutive subsets that contain only a single node. Let  $\mathcal{H}(CJ) = \{x \in \mathcal{P} \mid x(CJ) = 1\}$  and assume that  $\mathcal{H}(CJ) \subseteq \mathcal{G}$  where  $\mathcal{G}$  is the proper face of  $\mathcal{P}$  containing all the points  $x \in \mathcal{P}$  that satisfy the equality

$$\sum_{e \in CJ} \alpha_e x_e + \sum_{e \notin CJ} \beta_e x_e = \xi. \quad (14)$$

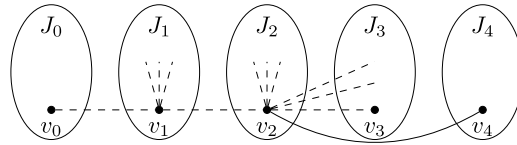
We will show that (14) is necessarily a linear combination of  $x(CJ) = 1$  and the equation  $x(E) = n - 1$  which implies that the considered circular-jump inequality defining the proper face  $\mathcal{H}(CJ)$  is facet defining (cf. Theorem 3.6 in [16]).

For  $\bar{\alpha} \in \mathbb{R}$  and  $\bar{\beta} \in \mathbb{R}$ , our proof builds upon the following four results that will be shown below:

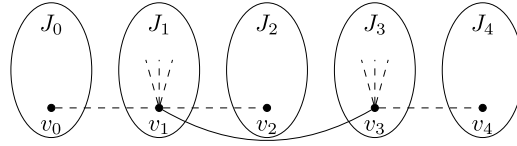
- Step 1:  $\beta_{u,v} = \bar{\beta}$ , for all  $\{u, v\} \in I(J_1) \cup I(J_2) \cup [J_1, J_2]$
- Step 2:  $\beta_{u,v} = \bar{\beta}$ , for all  $\{u, v\} \in [J_2, J_3] \cup I(J_3)$
- Step 3:  $\beta_{u,v} = \bar{\beta}$ , for all  $\{u, v\} \in [J_0, J_1] \cup [J_3, J_4] \cup [J_0, J_4]$
- Step 4:  $\alpha_{u,v} = \bar{\alpha}$ , for all  $\{u, v\} \in CJ$ .

Thus, evaluating the left-hand side of (14) at any incidence vector of a solution contained in  $\mathcal{H}(CJ)$  yields  $\xi = \bar{\alpha} + (n - 2)\bar{\beta}$ . Finally, the desired linear combination of  $x(CJ) = 1$  and  $x(E) = n - 1$  is obtained by multiplying these two equations with  $\bar{\alpha} - \bar{\beta}$  and  $\bar{\beta}$ , respectively, and summing them up to obtain the equality (14).

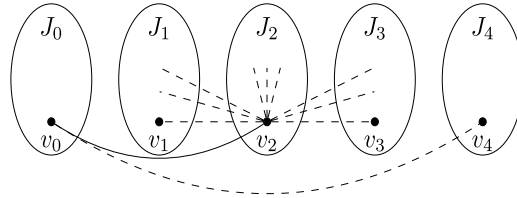
**Step 1:** Let  $\beta_{v_1, v_2} = \bar{\beta}$  for some  $\bar{\beta} \in \mathbb{R}$ . If  $|J_1| = |J_2| = 1$  the claim holds since  $\{v_1, v_2\} = I(J_1) \cup I(J_2) \cup [J_1, J_2]$ .



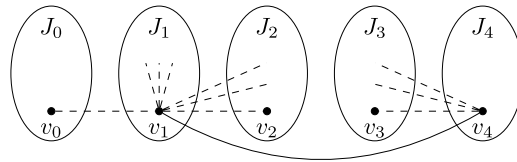
**Fig. 5.** Solution  $T^{2,4}$  with edge set  $E(T^{2,4}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_4\}\} \cup I(v_1) \cup I(v_2) \cup F(v_2)$ .



**Fig. 6.** Solution  $T^{1,3}$  with edge set  $E(T^{1,3}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}\} \cup I(v_1) \cup I(v_3)$ .



**Fig. 7.** Solution  $T^{0,2}$  with edge set  $E(T^{0,2}) = \{\{v_0, v_2\}, \{v_0, v_4\}\} \cup B(v_2) \cup I(v_2) \cup F(v_2)$ .



**Fig. 8.** Solution  $T^{1,4}$  with edge set  $E(T^{1,4}) = \{\{v_0, v_1\}, \{v_1, v_4\}\} \cup I(v_1) \cup F(v_1) \cup B(v_4)$ .

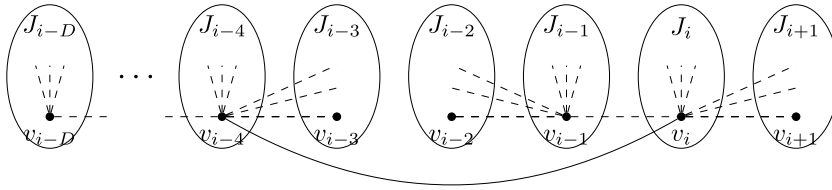
Thus, assume  $|J_1| > 1$  or  $|J_2| > 1$ . If  $|J_2| > 1$ , we consider the solutions  $T^{2,4}$  shown in Fig. 5 and  $T_u^{2,4}$ ,  $u \in J_2 \setminus \{v_2\}$ , obtained from  $T^{2,4}$  by replacing edge  $\{v_2, u\}$  by  $\{v_1, u\}$ . The incidence vectors of  $T^{2,4}$  and  $T_u^{2,4}$  satisfy  $\chi(CJ) = 1$  and hence also Eq. (14). Combining the two equations obtained from (14), we obtain  $\beta_{v_2, u} = \beta_{v_1, u}$ , for all  $u \in J_2 \setminus \{v_2\}$ . If  $|J_1| > 1$  we obtain  $\beta_{v_1, w} = \beta_{v_2, w}$ , for all  $w \in J_1 \setminus \{v_1\}$  in a similar way by using solutions  $T_w^{2,4}$ ,  $w \in J_1$  that are created from  $T^{2,4}$  by replacing edge  $\{v_1, w\}$  with  $\{v_2, w\}$ .

These steps are repeated starting with initial solutions where different centers  $v'_1 \in J_1$ ,  $v'_1 \neq v_1$ , and  $v'_2 \in J_2$ ,  $v'_2 \neq v_2$ , are chosen (if they exist, i.e., if the corresponding set has at least two nodes). Since, by assumption, either  $|J_1| > 1$  or  $|J_2| > 1$ , by using the fact that  $\beta_{v_1, v_2} = \bar{\beta}$ , we obtain  $\beta_{u, v} = \bar{\beta}$ , for all  $\{u, v\} \in I(J_1) \cup I(J_2) \cup [J_1, J_2]$ .

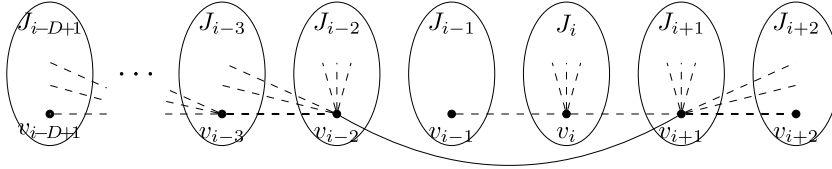
**Step 2:** We first observe that  $\beta_{u, v} = \rho$ , for all  $\{u, v\} \in [J_2, J_3] \cup I(J_3)$ , can be shown using analogous arguments as in Step 1 (initially starting with a solution where the roles of  $J_0$  and  $J_4$  as well as  $J_1$  and  $J_3$  are interchanged). Thus, if  $|J_2| > 1$ ,  $\beta_{u, v} = \bar{\beta}$ , for all  $\{u, v\} \in [J_2, J_3] \cup I(J_3)$  follows. Else, we obtain this result after additionally showing that  $\beta_{v_1, v_2} = \beta_{v_2, v_3}$ . The latter follows from combining the two equations obtained from Eq. (14) using the incidence vector of  $T^{1,3}$  (see Fig. 6) and the solution obtained from  $T^{1,3}$  by replacing edge  $\{v_1, v_2\}$  by  $\{v_2, v_3\}$ .

**Step 3:** To see that  $\beta_{uv} = \bar{\beta}$ , for all  $\{u, v\} \in [J_0, J_1]$ , we consider the solution  $T^{0,2}$  given in Fig. 7 and observe that another feasible solution can be constructed from  $T^{0,2}$  by replacing edge  $\{v_2, u\}$  by  $\{v_0, u\}$  for all  $u \in J_1$ . Since  $J_0 = \{v_0\}$  the result follows (from Step 1). Starting from a solution  $T^{2,4}$  with jump edge  $\{v_2, v_4\}$ ,  $\beta_{uv} = \bar{\beta}$ , for all  $\{u, v\} \in [J_3, J_4]$ , follows by analogous arguments. Finally,  $\beta_{v_0, v_4} = \bar{\beta}$  is obtained by combining the two equations obtained from (14) using the incidence vectors of  $T^{1,4}$  (see Fig. 8) and the solution obtained from  $T^{1,4}$  by replacing edge  $\{v_0, v_1\}$  by  $\{v_0, v_4\}$  (both solutions are in  $\mathcal{H}(CJ)$  since  $\{v_0, v_4\}$  is not a jump edge).

**Step 4:** We now show that the coefficients of all jump edges  $\{u, v\} \in CJ$  are identical, i.e.,  $\alpha_{u, v} = \bar{\alpha}$ , for all  $\{u, v\} \in CJ$ . Let  $\alpha_{v_0, v_2} = \bar{\alpha}$ . Combining the equations obtained from (14) for the characteristic vectors of  $T^{0,2}$  and  $T^{1,3}$ , it follows that  $\alpha_{v_1, v_3} = \bar{\alpha}$ . Replacing  $T^{1,3}$  by  $T^{2,4}$  or  $T^{1,4}$ , respectively, yields  $\alpha_{v_2, v_4} = \bar{\alpha}$  and  $\alpha_{v_1, v_4} = \bar{\alpha}$ . By constructing a solution using jump edge  $\{v_0, v_3\}$  (which is easy) we also obtain  $\alpha_{v_0, v_3} = \bar{\alpha}$ . Finally, by varying the chosen center nodes in the different subsets, and repeating these steps we obtain  $\alpha_{uv} = \bar{\alpha}$ , for all  $\{u, v\} \in CJ$ .



**Fig. 9.** Solution  $T^{i-4,i}$  with edge set  $E(T^{i-4,i}) = \bigcup_{j=i-D}^{i-4} (I(v_j) \cup \{\{v_j, v_{j+1}\}\}) \cup F(v_{i-4}) \cup \{\{v_{i-4}, v_i\}, \{v_{i-1}, v_i\}\} \cup B(v_{i-1}) \cup I(v_{i-1}) \cup I(v_i) \cup F(v_i)$ .



**Fig. 10.** Solution  $T^{i-2,i+1}$  with edge set  $E(T^{i-2,i+1}) = \bigcup_{j=i-D+2}^{i-2} B(v_j) \cup I(v_{i-2}) \cup \{\{v_{i-2}, v_{i+1}\}, \{v_{i-1}, v_i\}, \{v_i, v_{i+1}\}\} \cup I(v_i) \cup I(v_{i+1}) \cup F(v_{i+1})$ .

**Proof of Theorem 4.** To see that circular-jump inequalities define facets of  $\mathcal{P}$  for  $D \geq 4$ , let  $\mathcal{H}(CJ) = \{x \in \mathcal{P} \mid x(CJ) = 1\}$  such that  $\mathcal{H}(CJ) \subseteq \mathcal{G}$  where  $\mathcal{G}$  is the proper face of  $\mathcal{P}$  containing all the points  $x \in \mathcal{P}$  that satisfy the equality

$$\sum_{e \in CJ} \alpha_e x_e + \sum_{e \notin CJ} \beta_e x_e = \xi. \quad (15)$$

We will show that (15) is necessarily a linear combination of  $x(CJ) = 1$  and the equation  $x(E) = n - 1$  which implies that the considered circular-jump inequality defining the proper face  $\mathcal{H}(CJ)$  is facet defining (cf. Theorem 3.6 in [16]).

Our proof follows the following steps:

- Step 1:  $\beta_{u,v} = v_i$ , for all  $\{u, v\} \in I(J_i)$ ,  $0 \leq i \leq D + 1$
- Step 2:  $\beta_{u,v} = v_i$ , for all  $\{u, v\} \in [J_{i-1}, J_i]$ ,  $0 \leq i \leq D + 1$
- Step 3:  $v_{i-1} = v_i$ ,  $0 \leq i \leq D + 1$
- Step 4:  $\alpha_{u,v} = \mu$ , for all  $\{u, v\} \in CJ$

From Steps 1–4, we obtain that  $\beta_{u,v} = v$ , for all  $\{u, v\} \in E \setminus CJ$  and  $\alpha_{u,v} = \mu$ , for all  $\{u, v\} \in CJ$ . After evaluating (15) at any incidence vector of a solution contained in  $\mathcal{H}(CJ)$ , we obtain  $\xi = \mu + (n - 2)v$ . Hence, by multiplying  $x(CJ) = 1$  with  $\mu - v$  and  $x(E) = n - 1$  with  $v$  and summing up the resulting equations, we obtain the desired linear combination to represent the equality (15).

**Step 1:** We show that  $\beta_{u,v} = v_i$ , for all  $\{u, v\} \in I(J_i)$ ,  $0 \leq i \leq D + 1$ , whenever such edges exist, i.e., if  $|J_i| > 1$ . We note, that if  $|J_i| = 1$ , this claim obviously holds. Thus, let  $|J_i| > 1$  (that is  $|J_i| \geq 3$ ) and consider the solution  $T^{i-4,i}$  given in Fig. 9 which is feasible and in  $\mathcal{H}(CJ)$  for any  $0 \leq i \leq D + 1$ . Now, let  $k, l \in J_i$ ,  $k \neq l$ , be two arbitrary nodes from  $J_i \setminus \{v_i\}$ , and let  $T_{k,l}^{i-4,i}$  be the solution in  $\mathcal{H}(CJ)$  obtained from  $T^{i-4,i}$  by replacing the edge  $\{v_i, l\}$  by  $\{k, l\}$ . The incidence vectors of  $T^{i-4,i}$  and  $T_{k,l}^{i-4,i}$  satisfy  $x(CJ) = 1$  and hence also Eq. (15). After subtracting the two equations obtained from (15), we obtain  $\beta_{v_i,l} = \beta_{k,l}$ , for all  $k, l \in J_i \setminus \{v_i\}$ ,  $k \neq l$ .

**Step 2:** We first consider the case when  $|J_i| > 1$ . To see that  $\beta_{u,v} = v_i$  for all  $\{u, v\} \in [J_{i-1}, J_i]$ , we again consider the solution  $T^{i-4,i}$  and observe that after replacing edge  $\{v_i, k\}$  by  $\{v_{i-1}, k\}$  for an arbitrary node  $k \in J_i \setminus \{v_i\}$  (which exists by assumption), we obtain another solution in  $\mathcal{H}(CJ)$ . Using the two equations obtained from (15), we obtain  $\beta_{v_{i-1},k} = \beta_{v_i,k} = v_i$  by using the result obtained in Step 1. Furthermore, by changing the center of solution  $T^{i-4,i}$  in set  $J_{i-1}$  to  $l \in J_{i-1} \setminus \{v_{i-1}\}$  and by using the same arguments we also obtain  $\beta_{l,k} = \beta_{v_i,k} = v_i$ , for all  $l \in J_{i-1} \setminus \{v_{i-1}\}$ .

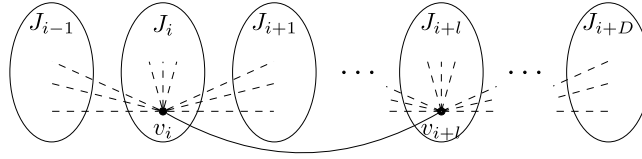
In case  $|J_i| = 1$ ,  $v_i$  has not been defined in Step 1. Thus, we set  $\beta_{v_{i-1},v_i} = v_i$ . Changing the center of solution  $T^{i-4,i}$  in set  $J_{i-1}$  to  $l \in J_{i-1} \setminus \{v_{i-1}\}$  we obtain  $\beta_{l,v_i} = \beta_{v_{i-1},v_i} = v_i$ , for all  $l \in J_{i-1} \setminus \{v_{i-1}\}$ .

**Step 3:** We show that  $v_{i-1} = v_i$  if  $|J_{i-1}| > 1$  by considering the solution  $T^{i-4,i}$  and another solution in  $\mathcal{H}(CJ)$  obtained by replacing edge  $\{v_{i-1}, k\}$  by  $\{v_i, k\}$  for an arbitrary node  $k \in J_{i-1} \setminus \{v_{i-1}\}$ . Using the two equations obtained from (15), we obtain  $v_{i-1} = \beta_{v_{i-1},k} = \beta_{v_i,k} = v_i$ .

For  $|J_{i-1}| = 1$  we consider the solution  $T^{i-2,i+1}$  given in Fig. 10 which is feasible and in  $\mathcal{H}(CJ)$  for any  $D \geq 4$  and  $0 \leq i \leq D + 1$ . Analogously to the previous steps, we can construct another solution in  $\mathcal{H}(CJ)$  by replacing edge  $\{v_{i-1}, v_i\}$  by edge  $\{v_{i-2}, v_{i-1}\}$  to obtain  $v_i = \beta_{v_{i-1},v_i} = \beta_{v_{i-2},v_{i-1}} = v_{i-1}$ .

**Step 4:** It finally remains to show that the coefficients of all jump edges are identical, i.e.,  $\alpha_{uv} = \mu$ , for all  $\{u, v\} \in CJ$ . To show this, consider the set of solutions  $\tilde{T}^{i,i+1}$  (see Fig. 11) showing that a solution in  $\mathcal{H}(CJ)$  exists using a jump arc between  $J_i$  and  $J_{i+1}$  for each relevant value of  $i$  and  $l$ , i.e.,  $0 \leq i \leq D + 1$ ,  $2 \leq l \leq \lceil \frac{D+1}{2} \rceil$ . Note that all cases with  $l > \lceil \frac{D+1}{2} \rceil$





**Fig. 11.** Solution  $\tilde{T}^{i,i+l}$ ,  $2 \leq l \leq \lceil \frac{D+1}{2} \rceil$ , with edge set  $E(\tilde{T}^{i,i+l}) = \{\{v_i, v_{i+l}\}\} \cup B(v_i) \cup I(v_i) \cup F(v_i) \cup \bigcup_{j=i+3}^{i+l} B(v_j) \cup I(v_{i+l}) \cup \bigcup_{j=i+l+1}^{i+D-1} F(v_j)$ .

are included since they yield a jump over less than  $\lceil \frac{D+1}{2} \rceil$  sets due to circularity. By systematically comparing any two of these solutions for all possible values of  $i$  and  $l$  and by doing this for all possible center nodes of subset  $J_i$  and  $J_{i+l}$ , respectively, we obtain the desired result.

### Appendix C. Facet proofs for generalized-circular-jump inequalities

**Proof of Theorem 5.** To see that generalized-circular-jump constraints define facets of  $\mathcal{P}$  if  $D \geq 4$  and  $1 \leq k \leq D-2$ , let  $P = (J_0, J_1, \dots, J_{D+k})$  be a non-trivial partition of  $V$  and let subsets  $C_\ell(P) \subset E$  be defined as above. Let  $\mathcal{H}(GCJ) = \{x \in \mathcal{P} \mid \sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) = k\} \subseteq \mathcal{G}$  be the set of feasible points that satisfy this particular generalized-circular-jump constraint with equality, where  $GCJ = \bigcup_{\ell=1}^k C_\ell(P)$  and  $\mathcal{G}$  is the proper face of  $\mathcal{P}$  containing all the points  $x \in \mathcal{P}$  that satisfy

$$\sum_{e \in GCJ} \alpha_e x_e + \sum_{e \notin GCJ} \beta_e x_e = \xi. \quad (16)$$

We will show that (16) is necessarily a linear combination of  $\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) = k$  and  $x(E) = n-1$  (cf. Theorem 3.6 in [16]).

By  $T_{k*1}^i$ , we denote a feasible solution that contains a chain of  $k$  consecutive jump edges from  $C_1(P)$ , starting from set  $J_i$  (see Fig. 12). Similarly,  $T_{1*\ell, (k-\ell)*1}^i$  will denote a feasible solution with a chain of jump edges starting at  $J_i$  such that the first jump edge is from  $C_\ell(P)$  (i.e., it jumps over  $\ell$  subsets), and it is followed by  $k-\ell$  jump edges from  $C_1(P)$ , see Fig. 13.

Our proof follows the following steps:

- Step 1:  $\beta_{u,v} = v$ , for all  $\{u, v\} \in E \setminus GCJ$
- Step 2:  $\alpha_{u,v} = \mu$ , for all  $\{u, v\} \in C_1(P)$
- Step 3:  $\alpha_{u,v} = \ell \cdot (\mu - v) + v$ , for all  $\{u, v\} \in C_\ell(P)$ ,  $2 \leq \ell \leq k$ .

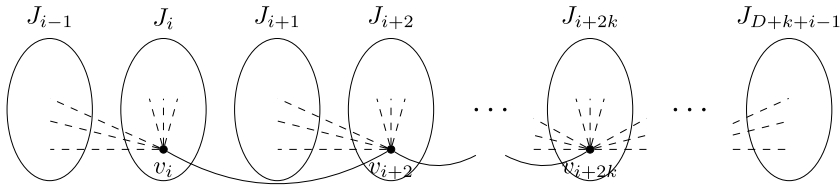
Finally, by evaluating a feasible solution from  $\mathcal{H}(GCJ)$  (that uses exactly one edge from  $C_k(P)$  skipping exactly  $k$  subsets and with  $n-2$  edges from  $E \setminus GCJ$ ) in (16), we obtain  $\xi = k \cdot (\mu - v) + (n-1)v$ . Hence, by multiplying  $\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) = k$  with  $\mu - v$  and  $x(E) = n-1$  by  $v$  and summing them up, we obtain a desired linear combination that results in Eq. (16).

**Step 1:** One can show that the coefficients of all edges  $e \in E \setminus GCJ$  are the same by the same technique used in the proof of Theorem 4 (by using slightly adapted solutions each using a single jump edge from set  $C_k(P)$ ). We therefore skip this part of the proof.

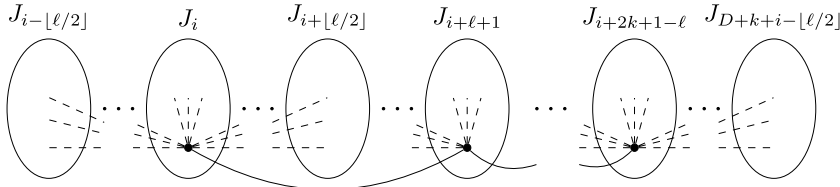
**Step 2:** To show that  $\alpha_{u,v} = \mu$ , for all  $\{u, v\} \in C_1(P)$ , we first observe that for any  $D \geq 4$ , any  $0 \leq \ell \leq D+k$  and  $2 \leq k \leq D-2$ , we can construct a feasible solution  $T_{k*1}^i$  using a sequence of  $k$  jump edges from  $C_1(P)$  starting from a node in subset  $i$  and which are adjacent to each other, see Fig. 12. The incidence vectors of  $T_{k*1}^i$  and  $T_{k*1}^{i+2}$  satisfy  $\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) = k$  and hence also Eq. (16). After subtracting the two equations obtained from (16) for these two solutions, we obtain  $\alpha_{v_i, v_{i+2}} = \alpha_{v_{i+2k}, v_{i+2(k+1)}}$ . Systematical repetition with solutions using different central nodes in sets  $J_i, J_{i+2}, \dots, J_{i+2k}$ , and  $J_{i+2(k+1)}$  yields  $\alpha_{uv} = \alpha_{st}$ ,  $u \in J_i, v \in J_{i+2}, s \in J_{i+2k}, t \in J_{i+2(k+1)}, 0 \leq i \leq D+k$ . If  $D+k$  is even (and thus the number of subsets is odd), we obtain  $\alpha_{uv} = \mu$ , for all  $\{u, v\} \in C_1(P)$  by performing the previous steps for any  $i \in \{0, 1, \dots, D+k\}$ . If  $D+k$  is odd, after considering each  $0 \leq i \leq D+k$ , we obtain  $\alpha_{uv} = \mu_1$ , for all  $\{u, v\} \in C_1(P)$  and  $u \in J_i$  with  $i \bmod 2 = 0$  and  $\alpha_{uv} = \mu_2$ , for all  $\{u, v\} \in C_1(P)$  and  $u \in J_i$  with  $i \bmod 2 = 1$ . By inserting the characteristic vectors of  $T_{k*1}^1$  and  $T_{k*1}^2$  into (16) and subtracting the results from each other we obtain  $k\mu_1 = k\mu_2$  and hence,  $\mu_1 = \mu_2 = \mu$  follows.

**Step 3:** We first consider jump edges  $\{u, v\} \in C_\ell(P)$  for  $1 \leq \ell < k$ , i.e., those that skip precisely  $\ell$  subsets. Observe first that coefficients for all  $e \in C_\ell(P)$  are the same, say  $\alpha^\ell$  for all  $\ell = 1, \dots, k$ , where  $\alpha^1 = \mu$ . This can be easily seen by comparing two different solutions that use one jump edge from  $C_\ell(P)$  and  $k-\ell$  edges from  $C_1(P)$ . Consider now a feasible solution  $T_{1*\ell, (k-\ell)*1}^i$  from  $\mathcal{H}(GCJ)$  using one jump edge from  $C_\ell(P)$  skipping exactly  $\ell$  subsets ( $2 \leq \ell < k$ ) and  $k-\ell$  jump edges from  $C_1(P)$  skipping one subset (see Fig. 13). Subtracting the two equations obtained from (16) for solutions  $T_{1*\ell, (k-\ell)*1}^i$  and  $T_{k*1}^i$  (using  $k$  edges from  $C_1(P)$ ) we obtain  $\alpha^\ell = \ell \cdot (\mu - v) + v$ , for all  $\{u, v\} \in C_\ell(P)$ ,  $1 \leq \ell < k$ .

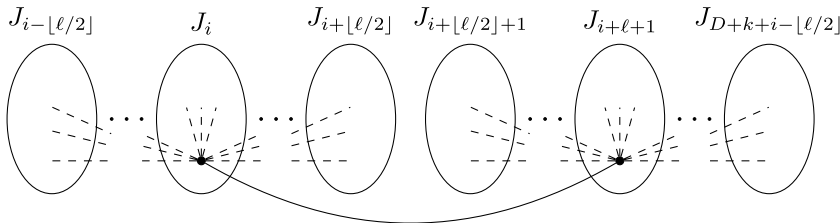
We also observe that the same argument suffices to show that  $\alpha_{uv} = k \cdot (\mu - v) + v$  for all edges  $\{u, v\} \in C_k(P)$  that skip precisely  $k$  subsets. To see that this equation also holds for those edges skipping more than  $k$  subsets, we observe that for each  $\{u, v\} \in C_k(P)$ , we can create a solution in  $\mathcal{H}(GCJ)$ , see Fig. 14. This exemplary solution is denoted by  $T_{1*\ell}^i$ , where, due



**Fig. 12.** Solution  $T_{k*1}^i$  with edge set  $E(T_{k*1}^i) = \bigcup_{j=0}^{k-1} (\{v_i+2j, v_i+2j+2\}) \cup I(v_i+2j) \cup B(v_i+2j) \cup I(v_i+2k) \cup B(v_i+2k) \cup \bigcup_{j=i+2k}^{D+k+i-2} F(v_j)$ .



**Fig. 13.** Solution  $T_{1*\ell, (k-\ell)*1}^i$  with edge set  $E(T_{1*\ell, (k-\ell)*1}^i) = \{v_i, v_i+\ell+1\} \cup I(v_i) \cup \bigcup_{j=0}^{k-\ell} (I(v_i+\ell+2j+1) \cup B(v_i+\ell+2j+1)) \cup \bigcup_{j=i-\lfloor \ell/2 \rfloor+1}^i B(v_j) \cup \bigcup_{j=i+\lfloor \ell/2 \rfloor+2}^{i+\ell} B(v_j) \cup \bigcup_{j=i}^{i+\lfloor \ell/2 \rfloor-1} F(v_j) \cup \bigcup_{j=i+2k+1-\ell}^{D+k+i-\lfloor \ell/2 \rfloor} F(v_j)$ .



**Fig. 14.** Solution  $T_{1*\ell}^i$  with edge set  $E(T_{1*\ell}^i) = \{v_i, v_i+\ell+1\} \cup I(v_i) \cup I(v_i+\ell+1) \cup \bigcup_{j=i-\lfloor \ell/2 \rfloor+1}^i B(v_j) \cup \bigcup_{j=i+\lfloor \ell/2 \rfloor+2}^{i+\ell+1} B(v_j) \cup \bigcup_{j=i}^{i+\lfloor \ell/2 \rfloor-1} F(v_j) \cup \bigcup_{j=i+\ell+1}^{D+k+i-\lfloor \ell/2 \rfloor-1} F(v_j)$ .

to circularity, it is sufficient to consider values of  $\ell$  such that  $k \leq \ell \leq \lceil \frac{D+k}{2} \rceil$ . We observe that  $T_{1*\ell}^i$  is a feasible solution, that contains two paths of length  $\lfloor \ell/2 \rfloor$  left and right from  $v_i$  and similarly, a path of length  $\lceil \ell/2 \rceil$  left of  $v_{i+\ell+1}$  and a path of length at most  $D - \lfloor \ell/2 \rfloor - 1$  on the right. Thus, the longest path in such constructed solution does not exceed  $D$ . Finally, the incidence vectors of  $T_{1*\ell}^i$  and a solution using  $k$  edges from  $C_1(P)$  (e.g.,  $T_{k*1}^i$ ) satisfy  $\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) = k$  and hence also Eq. (16). After subtracting the associated equations obtained from (16), we obtain the desired result.

#### Appendix D. Facet proof for cut generalized-circular-jump inequalities

**Proof of Theorem 6.** To show that cut generalized-circular-jump inequalities (cgj) define facets of  $\mathcal{P}$  for  $4 \leq D \leq n-k-2$ ,  $1 \leq k \leq D-3$ , let  $(R, S)$ ,  $|R| \geq D+k+1$ , be a non-trivial partition of the set of nodes  $V$ ,  $P = (J_0, J_1, \dots, J_{D+k})$  be a non-trivial partition of  $R$ , let subsets  $C_\ell(P) \subset E$  be defined as above,  $GCJ = \bigcup_{\ell=1}^k C_\ell(P)$ , let  $\mathcal{H}(GCJ, S) = \{x \in \mathcal{P} \mid \sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) + k \cdot x(R, S) + k \cdot x(E(S)) = k \cdot (|S| + 1)\}$  and let  $\mathcal{H}(GCJ, S) \subseteq \mathcal{G}$ , where  $\mathcal{G}$  is the proper face of  $\mathcal{P}$  containing all the points  $x \in \mathcal{P}$  that satisfy

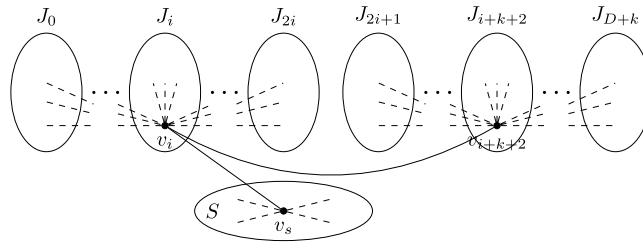
$$\sum_{e \in GCJ} \alpha_e x_e + \sum_{e \in E(R) \setminus GCJ} \beta_e x_e + \sum_{e \in [R, S]} \gamma_e x_e + \sum_{e \in E(S)} \delta_e x_e = \xi. \quad (17)$$

We will show that (17) is necessarily a linear combination of  $\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) + k \cdot x(R, S) + k \cdot x(E(S)) = k \cdot (|S| + 1)$  and  $x(E) = n - 1$  (cf. Theorem 3.6 from [16]).

Our proof follows the following steps:

- Step 1:  $\gamma_{u,v} = \mu'$ , for all  $\{u, v\} \in [R, S]$  and  $\delta_{u,v} = \mu'$ , for all  $\{u, v\} \in E(S)$
- Step 2:  $\beta_{u,v} = v$ , for all  $\{u, v\} \in E(R) \setminus GCJ$ ,  $\alpha_{u,v} = \mu$ , for all  $\{u, v\} \in C_1(P)$ , and  $\alpha_{u,v} = \ell \cdot (\mu - v) + v$ , for all  $\{u, v\} \in C_\ell(P)$ ,  $2 \leq \ell \leq k$
- Step 3:  $\mu' = k \cdot (\mu - v) + v$

Evaluating a solution from  $\mathcal{H}(GCJ, S)$  which consists of one edge from  $C_k(P)$ ,  $|S|$  edges from  $[R, S]$  and  $n - 2 - |S|$  edges from  $E(R) \setminus GCJ$ , we conclude that  $\xi = (k(\mu - v) + v)(|S| + 1) + (n - 2 - |S|)v = k(\mu - v)(|S| + 1) + (n - 1)v$ . Hence, by



**Fig. 15.** A feasible solution for  $D \geq 4$ ,  $2 \leq k \leq D-3$ ,  $\lceil k/2 \rceil \leq i \leq \lfloor D/2 \rfloor$ , with edge set  $\{\{v_i, v_s\}, \{v_i, v_{i+k+2}\}\} \cup I(v_s) \cup I(v_i) \cup I(v_{i+k+2}) \cup \bigcup_{j=1}^i B(v_j) \cup \bigcup_{j=2i+2}^{i+k+2} B(v_j) \cup \bigcup_{j=i}^{2i-1} F(v_j) \cup \bigcup_{j=i+k+2}^{D+k-1} F(v_j)$ .

multiplying  $\sum_{\ell=1}^k \ell \cdot x(C_\ell(P)) + k \cdot x(R, S) + k \cdot x(E(S)) = k \cdot (|S| + 1)$  with  $\mu - v$  and  $x(E) = n - 1$  by  $v$  and summing them up, we obtain a desired linear combination that results in Eq. (17).

**Step 1:** We will first show that  $\gamma_{v_i, u} = \delta_{v_s, u}$  for  $v_i \in J_i$ ,  $v_s \in S$ , and for all  $u \in S \setminus \{v_s\}$ . Consider the solution in  $\mathcal{H}(GCJ, S)$  given in Fig. 15. Another solution in  $\mathcal{H}(GCJ, S)$  is constructed by replacing edge  $\{v_s, u\}$  by  $\{v_i, u\}$  for some  $u \in S \setminus \{v_s\}$ . Combining the equations obtained from (17) for these solutions, we obtain  $\gamma_{v_i, u} = \delta_{v_s, u}$  for  $v_i \in J_i$ ,  $v_s \in S$ , and for all  $u \in S \setminus \{v_s\}$ . Repeating the same procedure using each  $v \in S$  as initial center of  $S$  (i.e., taking the role of  $v_s$ ), we obtain  $\gamma_{u, v} = \mu'$ , for all  $\{u, v\} \in E(S)$ , and by varying the center of  $J_i$  we also obtain  $\gamma_{u, v} = \mu'$ , for all  $u \in J_i$ , for all  $v \in S$ . Finally, repetition for  $i = 0, 1, \dots, D+k$ , yields  $\gamma_{u, v} = \mu'$ , for all  $\{u, v\} \in [R, S]$  and  $\delta_{u, v} = \mu'$ , for all  $\{u, v\} \in E(S)$ .

**Step 2:** Using the result of Step 1, these relations can be shown by repeating the necessary steps from the proof of Theorem 5 while directly connecting all nodes from  $S$  to a non-leaf node from  $R$  incident to at least one jump edge. Since it is easy to see that such a node always exists we skip the details.

**Step 3:** To show that  $\mu' = k \cdot (\mu - v) + v$  consider the solution in  $\mathcal{H}(GCJ, S)$  given in Fig. 15 and observe that  $\{v_i, v_{i+k+2}\} \in C_k(P)$  and that for  $1 \leq k \leq D-3$  (which is true by assumption of the theorem), another feasible solution from  $\mathcal{H}(GCJ, S)$  is obtained by replacing jump edge  $\{v_i, v_{i+k+2}\}$  by edge  $\{v_s, v_{i+k+2}\}$ . The result follows by comparing the equations obtained from (17) for these solutions.

## References

- [1] A. Balakrishnan, K. Altinkemer, Using a hop-constrained model to generate alternative communication network design, *ORSA J. Comput.* 4 (2) (1992) 192–205.
- [2] F. Bendali, I. Diarassouba, A.R. Mahjoub, J. Mailfert, The  $k$ -edge-disjoint 3-hop constrained paths polytope, *Discrete Optim.* 7 (4) (2010) 222–233.
- [3] G. Dahl, Notes on polyhedra associated with hop-constrained paths, *Oper. Res. Lett.* 25 (1999) 97–101.
- [4] G. Dahl, T. Flatberg, N. Foldness, L. Gouveia, Hop-constrained spanning trees: the jump formulation and a relax-and-cut method, Technical report, University of Oslo, Centre of Mathematics for Applications (CMA), Oslo, Norway, 2005.
- [5] G. Dahl, L. Gouveia, On the directed hop-constrained shortest path problem, *Oper. Res. Lett.* 32 (2004) 15–22.
- [6] G. Dahl, L. Gouveia, C. Requejo, On formulations and methods for the hop-constrained minimum spanning tree problem, in: M.G.C. Resende, P.M. Pardalos (Eds.), *Handbook of Optimization in Telecommunications*, Springer, 2006, pp. 493–515.
- [7] G. Dahl, D. Huygens, A.R. Mahjoub, P. Pesneau, On the  $k$ -edge-disjoint 2-hop constrained paths polytope, *Oper. Res. Lett.* (5) (2006) 577–582.
- [8] G.B. Dantzig, D.R. Fulkerson, S.M. Johnson, Solution of a large-scale traveling-salesman problem, *Oper. Res.* 2 (1954) 393–410.
- [9] I. Diarassouba, H. Kutucu, A.R. Mahjoub, Two node-disjoint 3-hop-constrained survivable network design and polyhedra, *Electron. Notes Discrete Math.* 41 (2013) 551–558.
- [10] M. Goemans, Y. Myung, A catalog of Steiner tree formulations, *Networks* 23 (1993) 19–23.
- [11] L. Gouveia, Minimal spanning trees with path constraints: formulations, valid inequalities and relaxations, Technical Report 11, Centro de Estatística e Aplicações da Universidade de Lisboa, 1993.
- [12] L. Gouveia, L. Simonetti, E. Uchoa, Modeling hop-constrained and diameter-constrained minimum spanning tree problems as Steiner tree problems over layered graphs, *Math. Program.* 128 (2011) 123–148.
- [13] D. Huygens, M. Labbé, A.R. Mahjoub, P. Pesneau, The two-edge connected hop-constrained network design problem: Valid inequalities and branch & cut, *Networks* 49 (2007) 116–133.
- [14] D. Huygens, A.R. Mahjoub, P. Pesneau, Two edge-disjoint hop-constrained paths and polyhedra, *SIAM J. Discrete Math.* 18 (2) (2004) 287–312.
- [15] T.L. Magnanti, L.A. Wolsey, Optimal trees, in: M. Ball, et al. (Eds.), *Handbooks in Operations Research and Management Science*, Vol. 7, Elsevier, 1995, pp. 503–615.
- [16] G.L. Nemhauser, L.A. Wolsey, *Integer and combinatorial optimization*, Wiley, New York, 1988.
- [17] M. Padberg, T.-Y. Sung, An analytical comparison of different formulations of the travelling salesman problem, *Math. Program.* 52 (1991) 315–357.
- [18] W.F. Riedl, A complete characterization of jump inequalities for the hop-constrained shortest path problem, *Discrete Appl. Math.* 225 (2017) 95–113.
- [19] K.A. Woolston, S.L. Albin, The design of centralized networks with reliability and availability constraints, *Comput. Oper. Res.* 15 (3) (1988) 207–217.